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M.A. / M.Sc. (First Semester)
Examination, Dec.-Jan., 2021-22
MATHEMATICS
Paper Second
(Real Analysis - I)

Time : Three Hours] ***[Maximum Marks : 80***

Note : All questions are compulsory.

SECTION-A
(Objective Type Questions)

1 each

Note:- All questions are compulsory.

1. The sequence $\{f_n\}_{n=1}^{\infty}$, where $f_n(x) = nx(1-x)^n$ is

   (a) converge uniformly on $[0, 1]$.

   (b) does not converge uniformly on $[0, 1]$. 0 is the point of non-uniformly convergence.

   (c) does not converge uniformly on $[0, 1]$. 1 is the point of non-uniformly convergence.

   (d) none of the above.

2. If $f_n(x) = n^2x(1-x)^n$, $x \in \mathbf{R}$ for each $n \in \mathbf{N}$. then

   (a) the limit function f is continuous

   (b) the $\{f_n(x)\}$ does not converge to f uniformly

   (c) Both (a) and (b)

   (d) none of the above.

3. The series $\frac{4}{\pi}\left[\sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \cdots\right]$ is converge uniformly in

   (a) $-\pi < x < \pi$

   (b) $-\pi < x \leq \pi$

   (c) $-\pi \leq x \leq \pi$

   (d) $-\pi \leq x < \pi$

4. The series $f(x) = \sum_{n=1}^{\infty} \frac{1}{n^3 + n^4x^2}$ is

   (a) converge uniformly for all $x \in \mathbf{R}$

   (b) does not converge uniformly for $x \in \mathbf{R}$, 0 is the point of non-uniformly convergence.

   (c) does not converge uniformly for $x \in \mathbf{R}$, 1 is the point of non-uniformly convergence.

   (d) none of the above.

5. Sum of the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$ is:

   (a) $\frac{3}{2}\log 2$

   (b) $\frac{1}{2}\log 2$

   (c) $\frac{2}{3}\log 2$

   (d) $\log 2$

6. The power series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^{2n-1}}{2n-1}$ is convergent for,

   (a) $-1 < x \leq 1$

   (b) $-1 < x < 1$

   (c) $0 < x \leq 1$

   (d) none of the above

7. The power series $1 + 2x + 3x^2 + 4x^3 + \cdots$ has radius of convergence equal to

   (a) e

   (b) 1

   (c) 2

   (d) none of the above

8. The power series $x + \frac{x^2}{2^2} + \frac{2!}{3^3}x^3 + \frac{3!}{4^4}x^4 + \cdots$ is

   (a) converges(absolutely) for all values of x.

   (b) does not converges for any values of x (other than 0).

   (c) converges(absolutely) for $|x| < e$

   (d) none of the above

9. Let $A \in L(X, Y)$ and $Ax = 0$ only when $x = 0$. then

(a) A is one-one
(b) A is onto
(c) both (a) and (b)
(d) none of the above

10. Let X, Y, Z be a vector space and let $A \in L(X, Y)$. $B \in L(Y, Z)$. then

(a) $BA \in L(X, Z)$
(b) A^{-1} is linear
(c) A^{-1} is invertible
(d) all of the above

11. Let $A, B \in L(R^n, R^m)$, then

(a) $||A + B|| \leq ||A|| + ||B||$
(b) $||A + B|| \geq ||A|| + ||B||$
(c) $||A + B|| = ||A|| + ||B||$
(d) none of the above

12. If $A \in L(R^n, R^m)$ and c is a scalar, then

(a) $||cA|| \leq |c|||A||$
(b) $||cA|| \geq |c|||A||$
(c) $||cA|| = |c|||A||$
(d) none of the above

13. If $x + y + z = u$; $y + z = uv$; $z = uvw$, then $\dfrac{\partial(x, y, z)}{\partial(u, v, w)}$ is:

(a) uv^2
(b) u^2v
(c) $-uv^2$
(d) $-u^2v$

14. Let to every $A \in L(R^n, R^1)$ corresponds a unique $y \in R^n$ Such that $Ax = x.y$, then

(a) $||A|| = |x|$
(b) $||A|| = |y|$
(c) $||A|| = |x||y|$
(d) none of the above

15. $\dfrac{\partial(u, v, w)}{\partial(x, y, z)} \cdot \dfrac{\partial(x, y, z)}{\partial(u, v, w)}$ is:

(a) 0
(b) 1
(c) -1
(d) none of the above

16. The function $f(x, y)$ is maximum at a stationary point if

(a) $(rt - s^2) > 0$ and $r > 0$
(b) $(rt - s^2) < 0$ and $r < 0$
(c) $(rt - s^2) > 0$ and $r < 0$
(d) none of the above.

17. An oriented 0−simplex σ is defined to be

(a) $+\mathbf{p}_0$
(b) $-\mathbf{p}_0$
(c) both (a) and (b)
(d) none of the above

18. The positively oriented boundary of Q^4 is:

(a) $[e_1, e_2, e_3, e_4] + [0, e_2, e_3, e_4] - [0, e_1, e_3, e_4] + [0, e_1, e_2, e_4] - [0, e_1, e_2, e_3]$
(b) $[e_1, e_2, e_3, e_4] - [0, e_2, e_3, e_4] - [0, e_1, e_3, e_4] + [0, e_1, e_2, e_4] + [0, e_1, e_2, e_3]$
(c) $[e_1, e_2, e_3, e_4] - [0, e_2, e_3, e_4] + [0, e_1, e_3, e_4] - [0, e_1, e_2, e_4] + [0, e_1, e_2, e_3]$
(d) $[e_1, e_2, e_3, e_4] - [0, e_2, e_3, e_4] + [0, e_1, e_3, e_4] + [0, e_1, e_2, e_4] - [0, e_1, e_2, e_3]$

19. Suppose σ_1 and σ_2 have the same set of vertices such that $\Gamma = \sigma_1 + \sigma_2 = 0$, then for all ω, $\int_\Gamma \omega$ is:

(a) 0
(b) 1
(c) 2
(d) none of the above

20. Let $\sigma = [\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2]$ than $\partial\sigma$ is equal to :

(a) $[\mathbf{p}_0, \mathbf{p}_2] + [\mathbf{p}_0, \mathbf{p}_1] + [\mathbf{p}_1, \mathbf{p}_2]$
(b) $[\mathbf{p}_0, \mathbf{p}_1] - [\mathbf{p}_2, \mathbf{p}_0] + [\mathbf{p}_1, \mathbf{p}_2]$
(c) $[\mathbf{p}_0, \mathbf{p}_1] + [\mathbf{p}_2, \mathbf{p}_0] + [\mathbf{p}_1, \mathbf{p}_2]$
(d) $[\mathbf{p}_0, \mathbf{p}_1] + [\mathbf{p}_2, \mathbf{p}_0] + [\mathbf{p}_2, \mathbf{p}_1]$

SECTION-B
(Very Short Answer Type Questions)

1.5 each

Note:- Attempt all questions

1. Write the Statement of Dirichlet's test.
2. Write the Statement of Weierstrass's M-test.
3. Define radius of convergence of power series.
4. Write the Statement of Abel's theorem for power series.
5. Define invertible linear operator.
6. Define continuously differentiable mapping.
7. What do you mean by stationary point for a function of several variables?
8. What do you mean by local maximum value for a real-valued function?
9. Define partition of unity.
10. Define basic $k-$forms.

SECTION-C
(Short Answer Type Questions)

2.5 each

Note:- Attempt all questions

1. Prove that the limit function of uniformly convergent sequence of continuous functions is itself continuous.

2. Test for term-by-term integration of the series $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ on [0.1] and show that

$$\int_0^1 \sum_{n=1}^{\infty} \frac{x^n}{n^2} dx = \sum_{n=1}^{\infty} \frac{1}{n^2(n+1)}.$$

3. Prove that the series obtained by differentiating a power series term by term has the same radius of convergence as the original series.

4. Show that

$$\frac{1}{2}(\tan^{-1} x)^2 = \frac{x^2}{2} - \frac{x^4}{4}\left(1 + \frac{1}{3}\right) + \frac{x^6}{6}\left(1 + \frac{1}{3} + \frac{1}{5}\right) - \cdots, \quad -1 < x \leq 1.$$

5. Let E be an open set in R^n, $\mathbf{f}$ maps E into R^m and $\mathbf{x} \in E$ and

$$\lim_{\mathbf{h} \to 0} \frac{|\mathbf{f}(\mathbf{x}+\mathbf{h}) - \mathbf{f}(\mathbf{x}) - A\mathbf{h}|}{|\mathbf{h}|} = 0,$$

holds with $A = A_1$ and with $A = A_2$. Then prove that $A_1 = A_2$.

6. If $A \in L(R^n, R^m)$ and $B \in L(R^m, R^k)$, then prove that $||BA|| \leq ||B||\ ||A||$.

7. If $y_1 = \cos x_1$, $y_2 = \sin x_1 \cos x_2$ and $y_3 = \sin x_1 \sin x_2 \cos x_3$, then find $\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)}$.

8. Describe the working rule of Lagrange's multiplier method for a function of several variables.

9. Let ω be $k-$forms of class of $\mathscr{C}''$ in some open set $E \in R^n$. Then prove that $d^2\omega = 0$. Here $d^2\omega$ means $d(d\omega)$.

10. Let ω and λ be $k-$forms and $m-$forms respectively of class $\mathscr{C}'$ in some open set $E \in R^n$. Then prove

$$d(\omega \wedge \lambda) = (d\omega) \wedge \lambda + (-1)^k\, \omega \wedge d\lambda.$$

SECTION-D
(Long Answer Type Questions)

5 each

Note:- Attempt all questions

1. State and prove Abel's Test.

Or

Let $f_n\{n = 1, 2, 3 \cdots\}$ be real functions defined on a set E in metric space and let the sequence $\{f_n\}$ converge uniformly to f on E. Let x_0 be a limit point of X, and suppose that

$$\lim_{x \to x_0} f_n(x) = A_n \quad (n = 1, 2, 3 \cdots)$$

then prove that

(i) the sequence $\{A_n\}$ or real constants converges , and

(ii) $\lim_{x \to x_0} f(x) = \lim_{n \to \infty} A_n$.

2. State and prove the Tauber's theorem on power series

Or

State and prove the Riemann's theorem on rearrangement of series.

3. Let Ω be the set of all invertible linear operator on R^n, then prove that,

(a) If $A \in \Omega$, $B \in L(R^n)$, and $||B - A||||A^{-1}|| < 1$ then $B \in \Omega$.

(b) Ω is open subset in $L(R^n)$ and the mapping $f : \Omega \to \Omega$ defined by $f(A) = A^{-1}$ for all $A \in \Omega$ is continuous.

Or

State and prove Taylor's theorem for a function of several variables.

4. Determine the maximum and minimum values of the function

$$f(x, y) = x^2 + y^2 + \frac{3\sqrt{3}}{2}xy$$

subject to the constraint $4x^2 + y^2 = 1$.

Or

Suppose T is a $\mathscr{C}'-$ mapping of an open set $E \subset R^n$ into an open set $V \subset R^m$, ϕ is a $k-$surface in E, and ω ia $k-$form in V. Then prove that

$$\int_{T\phi} \omega = \int_{\phi} \omega_T.$$